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LEGENDRE-TAU APPROXIMATION FOR
FUNCTIONAL DIFFERENTIAL EQUATIONS

Part III: Eigenvalue Approximations
and Uniform Stability

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LEGENDRE-TAU APPROXIMATION FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

PART III: EIGENVALUE APPROXIMATIONS AND UNIFORM STABILITY

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Abstract

The stability and convergence properties of the Legendre-tau approximation for hereditary differential systems are analyzed. We derive a characteristic equation for the eigenvalues of the resulting approximate system. As a result of this derivation we are able to establish that the uniform exponential stability of the solution semigroup is preserved under approximation. It is the key to obtaining the convergence of approximate solutions of the algebraic Riccati equation in trace norm.

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Introduction

It has been demonstrated in [6], [7] that the Legendre-tau approximation is a quite powerful approximation method for hereditary differential systems in many instances. However, there remained an important question which had not been resolved. This is the question concerned with the presentation of exponential stability under approximation. We establish it in Section 5.

As observed in [7], the Legendre-tau approximation scheme provides a good approximation technique for computation of eigenvalues for hereditary differential system as well as its optimal closed-loop system. We give a justification of this observation deriving a characteristic equation for eigenvalues of the approximating system and relating it to the Padé approximations of the exponential function. Moreover, it leads to a characterization of detectability and stabilizability conditions for the approximating system and the preservation of those properties under approximation.

The results discussed in this paper are similar to those for the 'averaging' approximation scheme [1] that have been obtained in [1], [5] and [9], and a great deal of our discussions are motivated and inspired by those investigations. We refer to [1], [8], [2], and [5] for the the summary of the earlier contributions on optimal control and numerical approximation problems for hereditary differential systems.

The following is a brief summary of the contents of this paper. In Section 2 we state the type of problems to be considered and review the equivalence results between hereditary differential equations and abstract Cauchy problems on the product space $\mathbb{R}^n \times L_2$, and then describe the Legendre-tau approximation scheme within the abstract framework. The convergence of

the adjoints semigroups $S^N(t)^*$ is proved in Theorem 2.2. In Section 3 we derive a characteristic equation for the approximating system and discuss its stability and convergence properties. In Section 4 we review results on the linear quadratic optimal control problem and prove the convergence of the solutions of the approximating evolutionary Riccati equation in trace norm. In Section 5 we prove the Gibson conjecture; i.e., that exponential stability is preserved under approximation. An important consequence of this is that by following the approach given in [5] one can establish the convergence of solutions of the approximating algebraic Riccati equation in trace norm (which implies strong and uniform convergence).

The notation used in this paper is standard and exactly the same as used in [6], [7]. We denote by Z the product space $\mathbb{R}^n \times L_2([-r, 0]; \mathbb{R}^n)$. Given an element $z \in Z$, $\eta \in \mathbb{R}^n$ and $\phi \in L_2$ denote the two coordinates of $z: z = (\eta, \phi)$. For any function ϕ of the independent variable θ , we shall use $\dot{\phi}$ or $\frac{\partial}{\partial \theta} \phi$ to denote the derivative of ϕ with respect to θ .

2. Legendre-Tau Approximation

In this paper we will restrict our analysis to the system:

$$\begin{aligned} \frac{d}{dt} x(t) &= A_0 x(t) + A_1 x(t-r) + \int_{-r}^0 A(\theta) x(t+\theta) d\theta + Bu(t) \\ x(0) &= \eta, \quad x(\theta) = \phi(\theta), \quad -r \leq \theta < 0, \end{aligned} \tag{2.1}$$

where $0 < r < +\infty$ and A_0 , A_1 and $A(\cdot)$ are $n \times n$ matrices, the elements of the latter being square integrable on $[-r, 0]$. It is well known [1] that for

Pages 3-4 missing from original

$$P^{N-1} \frac{\partial}{\partial t} z^N(t, \theta) = \frac{\partial}{\partial t} z^N(t, \theta)$$

$$\frac{d}{dt} z^N(t, 0) = A_0 z^N(t, 0) + A_1 z^N(t, -r) + \int_{-r}^0 A^N(\theta) z^N(t, \theta) d\theta + Bu(t),$$

where $A^N(\theta) = (P^{N-1}A)(\theta)$, $-r \leq \theta \leq 0$. As shown in [7], using (2.3), $z^N(t)$, $t \geq 0$ satisfies

$$\frac{\partial}{\partial t} z^N(t) = A^N z^N(t) + Bu(t) \quad (2.4)$$

$$z^N(0) = Q^N(n, \phi)$$

where

$$A^N = Q^N A L^N, \quad N \geq 1.$$

If we denote $S^N(t) = e^{A^N t}$, $t \geq 0$, then it is proved in [7] by using the Trotter-Kato semigroup approximation theorem [10], that $S^N(t)z$ converges strongly to $S(t)z$ for $t \geq 0$ and $z \in Z$, and the convergence is uniform in t on bounded intervals.

Lemma 2.1. For $(y, \psi) \in Z$,

$$(A^N)^*(y, \psi) = (A_0^T y + \psi^N(0), -\dot{\psi}^N + A^N(\cdot)^T y) \in Z,$$

where

$$\psi^N = P^{N-1} \psi + (A_1^T y - (P^{N-1} \dot{\psi})(-r)) p_N.$$

Proof: Note that for $(\eta, \phi) \in Z$,

$$A^N(\eta, \phi) = (A_0\eta + A_1\phi^N(-r) + \int_{-r}^0 A^N(\theta)\phi(\theta)d\theta, \dot{\phi}^N) \quad (2.5)$$

where

$$\phi^N = P^{N-1}\phi + (\eta - (P^{N-1}\phi)(0)) P_N.$$

Then we have

$$\begin{aligned} I &= \langle A^N(\eta, \phi), (y, \psi) \rangle_Z \\ &= \langle A_0\eta + A_1\phi^N(-r) + \int_{-r}^0 A^N(\theta)\phi(\theta)d\theta, y \rangle \\ &\quad + \int_{-r}^0 \langle \dot{\phi}^N(\theta), \psi(\theta) \rangle d\theta \\ &= \langle \eta, A_0^T y \rangle + \int_{-r}^0 \langle \phi(\theta), A^N(\theta)^T y \rangle d\theta \\ &\quad + \langle \phi^N(-r), A_1^T y \rangle + \langle \eta, \psi^N(\theta) \rangle - \langle \phi^N(-r), \psi^N(-r) \rangle \\ &\quad - \int_{-r}^0 \langle \phi^N(\theta), \dot{\psi}^N(\theta) \rangle d\theta, \end{aligned}$$

where $\phi^N(0) = \eta$ and $\dot{\phi}^N$ is a vector in \mathbb{R}^n whose elements are polynomials of degree less than $N-1$. Note that $\psi^N(-r) = A_1^T y$ and $P^{N-1}\dot{\psi}^N = \dot{\psi}^N$. Thus,

$$I = \langle \eta, A_0^T y + \psi^N(-r) \rangle + \int_{-r}^0 \langle \phi(\theta), A^N(\theta)^T y - \dot{\psi}^N(\theta) \rangle d\theta,$$

for all $(\eta, \phi), (y, \psi) \in Z$, which completes the proof.

(Q.E.D)

Although we will not pursue the details here, the error estimates in [6], [7] can be used to prove that

$$(A^N)^*(y, \psi) \longrightarrow A^*(y, \psi), \quad \text{for all } (y, \psi) \in \mathcal{D}((A^*)^2).$$

Hence the Trotter-Kato theorem again applies to obtain

Theorem 2.2:

$$S^N(t)^* z \longrightarrow S^*(t)z, \quad t \geq 0$$

uniformly on bounded t -intervals.

3. Characteristic Equations

In this section we will derive a characteristic equation for A^N . If $z^N \in Z$ is an eigenfunction of A^N :

$$(\lambda I - A^N)z^N = 0 \quad \text{for some complex number } \lambda, \quad (3.1)$$

then $z^N = Q^N z^N = (\eta^N, \sum_{k=0}^{N-1} a_k^N p_k)$. Let $L^N z = (\eta^N, \phi^N) \in \mathcal{D}(A)$ where

$$\phi^N = \sum_{k=0}^{N-1} a_k^N p_k + a_N^N p_N$$

$$a_N^N = \eta^N - \sum_{k=0}^{N-1} a_k^N.$$

Then from (3.1) and the definition (2.5) of A^N , we have

$$\lambda \eta^N - A_0 \eta^N - A_1 \phi^N(-r) - \int_{-r}^0 A^N(\theta) \phi^N(\theta) d\theta = 0 \quad (3.2)$$

$$\lambda(P^{N-1}\phi^N) - \dot{\phi}^N = 0. \quad (3.3)$$

From (3.3)

$$\lambda \begin{bmatrix} a_0^N \\ \vdots \\ a_{N-1}^N \end{bmatrix} - \frac{2}{r} S \begin{bmatrix} a_0^N \\ \vdots \\ a_N^N \end{bmatrix} = 0 \quad (3.4)$$

where S is the matrix representation of the derivative $(\partial/\partial\theta)$ and is given by

$$S = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 3 & 0 & 3 & \cdots & 0 & 3 \\ 0 & 0 & 0 & 5 & 0 & \cdots & 5 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2N-3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2N-1 \end{bmatrix}$$

for N even. For N odd, only the last column of S is different. Let us define $\{b_k^N\}_{k=0}^N$ by

$$b_k^N - b_{k+2}^N = a_k^N, \quad 0 \leq k \leq N-2$$

$$b_{N-1}^N = a_{N-1}^N \quad \text{and} \quad b_N^N = a_N^N.$$

Then from (3.4) if $\mu = (\frac{r}{2})\lambda$,

$$\mu b_0^N \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = E^N(\mu) \begin{bmatrix} b_1^N \\ \vdots \\ b_N^N \end{bmatrix}$$

where E^N is a tridiagonal matrix:

$$E^N(\mu) = \begin{bmatrix} 1 & \mu & & & & \\ -\mu & 3 & & & & \\ & -\mu & 5 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \mu \\ & & & & -\mu & 2N-1 \end{bmatrix}.$$

Applying Cramer's rule, one obtains

$$b_k^N = \mu^k \gamma_{k+1}^N b_0^N / \gamma_1^N$$

where

$$\gamma_k^N = \det \begin{bmatrix} 2k-1 & \mu & & & & \\ -\mu & & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \mu & \\ & & & -\mu & 2N-1 \end{bmatrix}, \quad 0 \leq k \leq N \quad \text{and} \quad \gamma_{N+1}^N = 1.$$

Since $\gamma_k^N = (2k-1)\gamma_{k+1}^N + \mu^2 \gamma_k^N$, it follows from (3.5) that

$$a_k^N = (2k+1)\mu^k \gamma_{k+1}^N b_0^N / \gamma_1^N, \quad 0 \leq k \leq N-1$$

$$a_N^N = \mu^N b_0^N / \gamma_1^N.$$

Note that $\eta^N = \sum_{k=0}^N a_k^N = b_0^N + b_1^N$. It now follows from (3.2) that

$$(\lambda I - A_0)(\gamma_1^N + \mu \gamma_2^N) - A_1(\gamma_1^N - \mu \gamma_2^N) - \int_{-r}^0 A^N(\theta) \xi^N(\theta) d\theta = 0,$$

where

$$\xi^N(\theta) = \sum_{k=0}^N (2k+1)\mu^k \gamma_{k+2}^N p_k\left(\frac{2\theta+r}{r}\right) b_0^N$$

and

$$\mu = \left(\frac{r}{2}\right)\lambda.$$

Thus, we obtain

$$\left(\lambda I - A_0 - A_1 \frac{\gamma_1^N - \mu \gamma_2^N}{\gamma_1^N + \mu \gamma_2^N} - \int_{-r}^0 A^N(\theta) \phi^N(\theta) d\theta \right) b_0^N = 0 \quad (3.6)$$

where

$$\phi^N(\theta) = \sum_{k=0}^N (2k+1)\mu^k \gamma_{k+2}^N / (\gamma_1^N + \mu \gamma_2^N) p_k\left(\frac{2\theta+r}{r}\right), \quad -r \leq \theta \leq 0.$$

For $N \geq 1$, let us define

$$\delta^N = \gamma_1^N \quad \text{and} \quad \beta^N = \gamma_2^N.$$

Then we have the recurrence formula:

$$\delta^N = (2N-1)\delta^{N-1} + \mu^2 \delta^{N-2} \quad (3.7)$$

$$\beta^N = (2N-1)\beta^{N-1} + \mu^2 \beta^{N-2}$$

with $\delta^0 = \delta^1 = 1$, $\beta^0 = 0$ and $\beta^1 = 1$, so that if $\rho_{\pm}^N = \delta^N \pm \mu\beta^N$, then the ρ 's satisfy the recurrence formula:

$$\rho^N = (2N-1)\rho^{N-1} + \mu^2 \rho^{N-2}$$

with $\rho_{\pm}^0 = 1$ and $\rho_{\pm}^1 = 1 \pm \mu$. The recurrence formula (3.7) is exactly the same as that for the diagonal Padé approximation of $\exp(-r\lambda)$ [4]. Note that

$$\phi^N(0) = 1 \quad \text{and} \quad \phi^N(-r) = \frac{\rho_{-}^N}{\rho_{+}^N} = \text{Padé}(e^{-r\lambda}). \quad (3.8)$$

Hence from (3.6), we obtain:

Theorem 3.1 λ is an eigenvalue of A^N if and only if λ satisfies
 $\det \Delta^N(\lambda) = 0$ where

$$\Delta^N(\lambda) = \lambda I - A_0 - A_1 \text{Padé}(e^{-r\lambda}) - \int_{-r}^0 A^N(\theta) \phi^N(\theta) d\theta.$$

Lemma 3.2: If the elements of $A(\cdot)$ are absolutely continuous on $[-r, 0]$,
then there exists a positive constant C such that

$$\left| \int_{-r}^0 A^N(\theta) \phi^N(\theta) d\theta \right| \leq C$$

for every $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq 0$.

Proof: Since ϕ^N satisfies

$$\lambda \phi^N - \dot{\phi}^N = \lambda a^N(\lambda) p_N$$

where $a^N(\lambda) = \mu^N / \rho_+^N$, we have

$$\phi^N(\theta) = e^{\lambda \theta} + \int_{\theta}^0 \lambda a^N(\lambda) e^{\lambda(\theta-\xi)} p_N\left(\frac{2\xi+r}{r}\right) d\xi.$$

Thus,

$$\begin{aligned} \int_{-r}^0 A^N(\theta) \phi^N(\theta) d\theta &= \int_{-r}^0 A(\theta) \phi^N(\theta) d\theta - \int_{-r}^0 A(\theta) a^N(\lambda) p_N d\theta \\ &= \int_{-r}^0 A(\theta) \left(e^{\lambda \theta} + \int_{\theta}^0 \lambda a^N(\lambda) e^{\lambda(\theta-\xi)} p_N d\xi - a^N(\lambda) p_N \left(\frac{2\theta+r}{r} \right) \right) d\theta. \end{aligned} \quad (3.9)$$

Here,

$$\begin{aligned} &\int_{-r}^0 A(\theta) \int_{\theta}^0 \lambda e^{\lambda(\theta-\xi)} p_N\left(\frac{2\xi+r}{r}\right) d\xi \\ &= \int_{-r}^0 \int_{-r}^{\xi} A(\theta) \lambda e^{\lambda(\theta-\xi)} d\theta p_N\left(\frac{2\xi+r}{r}\right) d\xi \\ &= \int_{-r}^0 \left(A(\xi) - A(-r) e^{-\lambda(r+\xi)} - \int_{-r}^{\xi} e^{\lambda(\theta-\xi)} \dot{A}(\theta) d\theta \right) p_N d\xi. \end{aligned}$$

Hence from (3.9)

$$\begin{aligned} \left| \int_{-r}^0 A^N(\theta) \phi^N(\theta) d\theta \right| &\leq \int_{-r}^0 |A(\theta)| d\theta + |a^N(\lambda)| \int_{-r}^0 \left(|A(-r)| + \int_{-r}^{\xi} |\dot{A}(\theta)| d\theta \right) d\xi \\ &\leq \int_{-r}^0 |A(\theta)| d\theta + r \left(|A(-r)| + \int_{-r}^0 |\dot{A}(\theta)| d\theta \right) \end{aligned}$$

on $\phi^+ = \{\lambda \in \phi, \operatorname{Re} \lambda \geq 0\}$ where we used $|a^N(\lambda)| \leq 1$ on ϕ^+ . (Q.E.D.)

Corollary 3.3: If the elements of $A(\cdot)$ are absolutely continuous on $[-r, 0]$, then the complex function $\det \Delta^N(\lambda)$ cannot have a zero in the closed right halfplane outside the disc of radius $|A_0| + |A_1| + C$.

Proof: Note that $|\operatorname{Pade}(e^{-r\lambda})| \leq 1$ on ϕ^+ . The corollary follows from Lemma 3.2. (Q.E.D.)

Lemma 3.4: $\Delta^N(\lambda)$ converges to $\Delta(\lambda)$ uniformly on every bounded subset of the complex plane.

Proof: We only need to show that

$$\int_{-r}^0 A^N(\theta) \phi^N(\theta) d\theta \longrightarrow \int_{-r}^0 A(\theta) e^{\lambda\theta} d\theta, \text{ uniformly.}$$

From (3.9)

$$\begin{aligned} &\left| \int_{-r}^0 A^N(\theta) \phi^N(\theta) d\theta - \int_{-r}^0 A(\theta) e^{\lambda\theta} d\theta \right| \\ &= \left| \int_{-r}^0 a^N(\lambda) A(\theta) \left(\int_{\theta}^0 \lambda e^{\lambda(\theta-\xi)} p_N d\xi - p_N \left(\frac{2\theta+r}{r} \right) \right) d\theta \right| \end{aligned}$$

$$\begin{aligned}
 &= \left| \int_{-r}^0 a^N(\lambda) \left[-A(\xi) + \int_{-r}^{\xi} \lambda e^{\lambda(\theta-\xi)} A(\theta) d\theta \right] p_N\left(\frac{2\xi+r}{r}\right) d\xi \right| \\
 &\leq |a^N(\lambda)| \left(1 + r \int_{-r}^0 |\lambda e^{\lambda\theta}|^2 d\theta \right)^{1/2} \left(\int_{-r}^0 |A(\theta)|^2 d\theta \right)^{1/2} \left(\frac{r}{2N+1} \right)^{1/2},
 \end{aligned}$$

which converges uniformly to zero on every bounded subset of the complex plane. (Q.E.D)

4. Riccati Equations

Let G be a non-negative, self-adjoint operator on Z and C be a $p \times n$ matrix. Consider the optimal control problem on a finite interval: for given initial data $z = (\eta, \phi) \in Z$

$$\text{minimize } J(u; [0, T]) = \int_0^T (|Cz(t)|^2 + |u(t)|^2) dt + \langle Gz(T), z(T) \rangle \quad (4.1)$$

over $u \in L_2([0, T]; \mathbb{R}^m)$ subject to (2.2), where $C(\eta, \phi) = C\eta$ for $(\eta, \phi) \in Z$. It is well known [3], [5] that the optimal solution u^0 to (4.1) is given by

$$u^0(t) = -B^* \Pi(t) z^0(t)$$

where $\Pi(t)$, $t \leq T$ is the unique non-negative, self-adjoint solution to the Riccati equation:

$$\frac{d}{dt} \langle \Pi(t) z, z \rangle = -2 \langle Az, \Pi(t) z \rangle + \langle B^* \Pi(t) z, B^* \Pi(t) z \rangle - \langle Cz, Cz \rangle \quad (4.2)$$

for all $z \in \mathcal{D}(A)$

$$\Pi(T) = G,$$

and $z^0(t)$ satisfies the evolution equation

$$\frac{d}{dt} z^0(t) = (A - BB^* \Pi(t)) z^0(t)$$

$$z^0(0) = z.$$

Consider the Nth approximate problem to (4.1): minimize

$$J^N(u; [0, T]) = \int_0^T (|Cz^N(t)|^2 + |u(t)|^2) dt + \langle G^N z^N(T), z^N(T) \rangle \quad (4.3)$$

subject to (2.4) where $G^N = Q^N G Q^N$. The optimal control u^N to (4.3) is given by

$$u^N(t) = -B^* \Pi^N(t) z^N(t),$$

where $\Pi^N(t)$, $t \leq T$ is the unique non-negative, self-adjoint operator to the Nth approximate Riccati equation:

$$\frac{d}{dt} \langle \Pi^N(t) z, z \rangle = -2 \langle A^N z, \Pi^N(t) z \rangle + \langle B^* \Pi^N(t) z, B^* \Pi^N(t) z \rangle - \langle Cz, Cz \rangle$$

for all $z \in Z$ (4.4)

$$\Pi^N(T) = G^N$$

and $z^N(t)$ satisfies

$$\frac{d}{dt} z^N(t) = (A^N - BB^* \Pi^N(t)) z^N(t), \quad t \geq 0$$

$$z^N(0) = Q^N z.$$

From Theorem 2.2 and Theorem 6.1 - 6.3 in [5], we have the desired convergence results:

Theorem 4.1. $\Pi^N(t)$ converges strongly to $\Pi(t)$ for $t \leq T$ and the convergence is uniform for t in bounded intervals. If $\hat{u}^N = -B^* \Pi^N(t) \hat{z}^N(t)$, $t \leq T$ where $\hat{z}^N(t)$ is the mild solution to

$$\frac{d}{dt} \hat{z}^N(t) = (A - BB^* \Pi^N(t)) \hat{z}^N(t)$$

$$\hat{z}^N(0) = z;$$

i.e., the N th feedback control law applied to the original hereditary system (2.1), then for any $\varepsilon > 0$ there exists a nondecreasing function $N_\varepsilon(\cdot): [0, \infty) \rightarrow \mathbb{R}^+$ such that for $N \geq N_\varepsilon(T)$

$$J(\hat{u}^N; [0, T]) \leq J(u^0; [0, T]) + \varepsilon \|z\|^2.$$

Moreover, if G is given by $G(\eta, \phi) = (G_0 \eta, 0)$, then $\Pi^N(t)$ converges in trace norm to $\Pi(t)$ for $t \leq T$ and the convergence is uniform for t in bounded intervals.

Let us now consider the optimal control problem on the infinite interval. For given initial data $z = (\eta, \phi) \in Z$, minimize the cost functional:

$$J(u, z) = \int_0^{\infty} (|Cz(t)|^2 + |u(t)|^2) dt \quad (4.5)$$

subject to (2.2).

Definition: The pair (A, B) is stabilizable if there exists an operator $K \in L(Z, \mathbb{R}^m)$ such that $A - BK$ generates a uniformly exponentially stable semigroup. (C, A) is detectable if (A^*, C^*) is stabilizable.

The following theorem is now standard [11], [5].

Theorem 4.2. If (A, B) is stabilizable and (C, A) is detectable, then the algebraic Riccati equation (ARE):

$$(A^* \Pi + \Pi A - \Pi B B^* \Pi + C^* C)z = 0 \quad \text{for all } z \in \mathcal{D}(A)$$

has a unique self-adjoint, non-negative solution. Moreover, if Π denotes the said solution, then $A - B B^* \Pi$ generates a uniformly exponentially stable semigroup and the optimal control to (4.5) is given by

$$u^0(t) = -B^* \Pi z^0(t)$$

where $z^0(t)$ is the mild solution to

$$\frac{d}{dt} z^0(t) = (A - B B^* \Pi) z^0(t)$$

$$z^0(0) = z$$

Remark: For the hereditary differential systems, we have

(i) (A, B) is stabilizable if and only if

$$\text{rank}[\Delta(\lambda), B] = n \quad \text{for all } \lambda \in \mathbb{C}^+,$$

(ii) (C, A) is detectable if and only if

$$\text{rank} \begin{bmatrix} \Delta(\lambda) \\ C \end{bmatrix} = n \quad \text{for all } \lambda \in \mathbb{C}^+$$

(iii) a complex number $\lambda \in \sigma(A - BB^* \Pi)$ if and only if $\text{Re } \lambda < 0$ and λ satisfies $\det \hat{\Delta}(\lambda) = 0$ where

$$\hat{\Delta}(\lambda) = \lambda I - \begin{bmatrix} D(\lambda) & -BB^T \\ -C^T C & -D(-\lambda)^T \end{bmatrix}$$

$$D(\lambda) = A_0 + A_1 e^{-r\lambda} + \int_{-r}^0 A(\theta) e^{\lambda\theta} d\theta [5].$$

The next lemma gives the characterization of stabilizing and detectability for the approximating system (2.4) which is precisely the analogon to those given in Remark.

Lemma 4.3.

(i) (A^N, B) is stabilizable if and only if

$$\text{rank}[\Delta^N(\lambda), B] = n \quad \text{for all } \lambda \in \mathbb{C}^+$$

(ii) (C, A^N) is detectable if and only if

$$\text{rank} \begin{bmatrix} \Delta^N(\lambda) \\ C \end{bmatrix} = n \quad \text{for all } \lambda \in \mathbb{C}^+.$$

Proof: We will only prove the statement (ii) since (i) follows from Lemma 2.1 and (ii) by duality. It follows from the finite dimensional linear system theory that (C, A^N) is detectable if and only if

$$A^N z = \lambda z \quad \text{and} \quad Cz = 0 \quad \text{for } \lambda \in \mathbb{C}^+$$

imply $Q^N z = 0$. Hence from (3.6), (3.8) and Theorem 3.1, it is equivalent to

$$\ker \Delta^N(\lambda) \cap \ker C = \{0\} \quad \text{for } \lambda \in \mathbb{C}^+. \quad (\text{Q.E.D.})$$

The next corollary follows from Corollary 3.3, Lemma 3.4 and Lemma 4.3.

Corollary 4.4. Suppose that the elements of $A(\cdot)$ are absolutely continuous on $[-r, 0]$. If (A, B) is stabilizable ((C, A) is detectable, respectively), then (A^N, B) is stabilizable ((C, A^N) is detectable, respectively) for N sufficiently large.

For the rest of this section we assume that the conditions stated in Corollary 4.4 hold. Let us consider the Nth approximate problem to (4.5):
minimize

$$J^N(u, z) = \int_0^\infty (|Cz^N(t)|^2 + |u(t)|^2) dt \quad (4.6)$$

subject to (2.4). The optimal control u^N to (4.6) is given by

$$u^N(t) = -B^* \Pi^N z^N(t)$$

and

$$z^N(t) = e^{(A^N - BB^* \Pi^N)t} Q^N z,$$

where Π^N is the unique nonnegative, self-adjoint solution to $(ARE)^N$:

$$(A^N)^* \Pi^N + \Pi^N A^N - \Pi^N BB^* \Pi^N + C^* C = \phi.$$

In the following lemma we give a characteristic equation for eigenvalues of $A^N - BB^* \Pi^N$.

Lemma 4.5: A complex number λ is an eigenvalue of $A^N - BB^* \Pi^N$ if and only if $\text{Re } \lambda < 0$ and λ satisfies $\det \hat{\Delta}^N(\lambda) = 0$ where

$$\hat{\Delta}^N(\lambda) = \lambda I - \begin{bmatrix} D^N(\lambda) & -BB^T \\ -C^T C & -D^N(-\lambda)^T \end{bmatrix}$$

$$D^N(\lambda) = A_0 + A_1 \phi^N(-r) + \int_{-r}^0 A^N(\theta) \phi^N(\theta) d\theta.$$

Proof. It follows from [5] that $\lambda \in \sigma(A^N - BB^* \Pi^N)$ if and only if $\operatorname{Re} \lambda < 0$ and $\lambda \in \sigma(H^N)$ where

$$H^N = \begin{bmatrix} A^N & -BB^* \\ -C^*C & -(A^N)^* \end{bmatrix} \quad \text{on } Z^N \times Z^N$$

and $Z^N \equiv Q^N Z$.

From (2.5) and Lemma 2.1, if $\lambda \in \sigma(H^N)$, then there exists an element $((\eta, \tilde{\phi}), (y, \tilde{\psi})) \in Z^N \times Z^N$ such that

$$A_0 \eta + A_1 \phi(-r) + \int_r^0 A^N(\theta) \phi(\theta) d\theta - BB^T y = \lambda \eta$$

$$\dot{\tilde{\phi}} = \lambda \tilde{\phi}$$

$$-C^T Cy - A_0^T y - \psi(0) = \lambda y$$

$$A^N(\cdot)^T y - \dot{\tilde{\psi}} = \lambda \tilde{\psi}$$

where

$$\phi = \tilde{\phi} + (\eta - \tilde{\phi}(0))p_N, \quad \eta = \phi(0)$$

$$\psi = \tilde{\psi} + (A_1^T y - \tilde{\psi}(-r))p_N \quad \text{and} \quad \psi(-r) = A_1^T y.$$

Thus the similar arguments as given in Section 3 allow us to conclude that

$\lambda \in \sigma(H^N)$ if and only if $\det \hat{\Delta}^N(\lambda) = 0$. (Q.E.D.)

5. Gibson's Conjecture

In this section, we prove the Conjecture 7.1 in [5] for the Legendre-tau approximation: if the semigroup $\{S(t), t \geq 0\}$ is uniformly, exponentially stable, then there exist positive constants M and w such that, for N sufficiently large

$$\|S^N(t)\| \leq M e^{-wt}.$$

To this end we need the following two results.

Lemma 5.1: Let us denote by A_0 , a generator on Z defined by $\mathcal{D}(A_0) = \mathcal{D}(A)$ and

$$A_0(\phi(0), \dot{\phi}) = (-\dot{\phi}(0), \ddot{\phi}) \in Z$$

and define $A_0^N = Q^N A_0 L^N$, $N \geq 1$. Then there exist positive constants ω_0 and M_0 such that

$$\|e^{A_0^N t}\| \leq M_0 e^{-\omega_0 t}$$

and

$$\int_0^\infty \|E e^{A_0^N t} z\|_{\mathbb{R}^m}^2 \leq M_0 \|z\|^2$$

where E is an operator on Z defined by

$$E(\eta, \phi) = \phi(-r) \quad \text{for } (\eta, \phi) \in Z.$$

Proof: We define the weighted inner product $\langle \cdot, \cdot \rangle_g$ on Z by

$$\langle (\eta, \phi), (y, \psi) \rangle_g = \langle \eta, y \rangle_{R^n} + \int_{-r}^0 \langle \phi(\theta), \psi(\theta) \rangle_{R^n} g(\theta) d\theta \quad (5.1)$$

where g is positive on $[-r, 0]$ defined by

$$g(\theta) = \frac{3}{4} + \frac{1}{4} \left(\frac{2\theta + r}{r} \right).$$

Since $\frac{1}{2} \leq g(\theta) \leq 1$ on $[-r, 0]$, if Z_g denotes the completion of Z with respect to the inner product (5.1) and $\| \cdot \|_g$ denotes the induced norm, then

$$\|z\|_g^2 \leq \|z\|^2 \leq 2\|z\|_g^2. \quad (5.2)$$

For $z^N = (\eta, \phi) \in Z^N = Q^N Z$

$$I = \langle A_0^N z^N, z^N \rangle_g = \int_{-r}^0 \langle P^{N-1} \phi^N, \dot{\phi}^N \rangle_{R^n} g(\theta) d\theta - |\eta|^2$$

where $\phi^N = \phi + (\eta - \phi(0))P_N$ and $\phi^N(0) = \eta$.

Note that

$$\begin{aligned} & \int_{-r}^0 \langle (\eta - \phi(0))P_N \left(\frac{2\theta + r}{r} \right), \dot{\phi}^N \rangle_{R^n} g(\theta) d\theta \\ &= \int_{-r}^0 \langle (\eta - \phi(0))P_N, \dot{\phi}^N \rangle_{R^n} \frac{1}{4} \left(\frac{2\theta + r}{r} \right) d\theta \end{aligned}$$

$$\begin{aligned}
 &= \int_{-r}^0 |\eta - \phi(0)|^2 P_N \dot{P}_N \frac{1}{4} \left(\frac{2\theta + r}{r} \right) d\theta \\
 &= \frac{1}{4} \int_{-1}^1 |\eta - \phi(0)|^2 P_N(x) \dot{P}_N(x) x \, dx \\
 &= \frac{1}{4} \int_{-1}^1 (2N-1) |\eta - \phi(0)|^2 P_N(x) P_{N-1}(x) dx \\
 &= \frac{1}{4} \int_{-1}^1 |\eta - \phi(0)|^2 P_N(x) (NP_N(x) - (N-1)P_{N-2}(x)) dx \\
 &= \frac{1}{4} \left(\frac{2N}{2N+1} \right) |\eta - \phi(0)|^2,
 \end{aligned}$$

where we used that $P^{N-1} \dot{\phi}^N = \dot{\phi}^N$ and

$NP_N(x) = (2N-1)xP_N(x) - (N-1)P_{N-2}(x)$, $-1 \leq x \leq 1$. Thus, we have

$$\begin{aligned}
 I &= \int_{-r}^0 \langle \phi^N, \dot{\phi}^N \rangle g(\theta) d\theta - |\eta|^2 - \frac{1}{4} \left(\frac{2N}{2N+1} \right) |\eta - \phi(0)|^2 \\
 &= \frac{1}{2} |\phi^N(0)|^2 - \frac{1}{4} |\phi^N(-r)|^2 - \frac{1}{8} \left(\frac{2}{r} \right) \int_{-r}^0 |\phi^N|^2 d\theta \\
 &\quad - |\eta|^2 - \frac{1}{4} \left(\frac{2N}{2N+1} \right) |\eta - \phi(0)|^2 \\
 &= -\frac{1}{2} |\eta|^2 - \frac{1}{4} |\phi^N(-r)|^2 - \frac{1}{8} \left(\frac{2}{r} \right) \int_{-r}^0 |\phi|^2 d\theta - \frac{1}{4} |\eta - \phi(0)|^2 \\
 &\leq -\omega_0 \|\eta, \phi\|_g^2 - \frac{1}{8} |\phi(-r)|^2,
 \end{aligned}$$

where $\omega_0 = \min(\frac{1}{2}, \frac{1}{4r})$. It now follows that

$$\left\langle \frac{d}{dt} z^N(t), z^N(t) \right\rangle_g \leq -\omega_0 \|z^N(t)\|_g^2 - \frac{1}{8} |\varepsilon z^N(t)|^2$$

or

$$\begin{aligned} \|z^N(t)\|_g^2 &\leq \|z^N(0)\|_g^2 - 2\omega_0 \int_0^t \|z^N(s)\|_g^2 ds - \frac{1}{4} \int_0^t |\bar{E} z^N(s)|^2 ds \\ &\leq \|z^N(0)\|_g^2 - 2\omega_0 \int_0^t \|z^N(s)\|_g^2 ds. \end{aligned} \quad (5.3)$$

Hence by Gronwall's lemma

$$\|z^N(t)\|_g \leq e^{-\omega_0 t} \|z^N(0)\|_g, \quad t \geq 0.$$

From (5.3), for all $t \geq 0$

$$\int_0^t |\bar{E} z^N(s)|^2 ds \leq 4\|z^N(0)\|_g^2 \leq 4\|z^N\|^2 \leq 4\|z\|^2. \quad (\text{Q.E.D.})$$

Corollary 5.2. Let I denote an operator on Z defined by $I(\eta, \phi) = \eta \in \mathbb{R}^n$ for $(\eta, \phi) \in Z$. Then, if

$$\int_0^\infty |I e^{Nt} z|^2 dt \leq K \|z\|^2$$

for some positive constant K , there exists positive constants ω_1 and M_1 , such that

$$\|e^{A^N t}\| \leq M_1 e^{-\omega_1 t}, \quad t \geq 0.$$

Proof: The same argument as in the proof of Lemma 5.1 yields that for $(\eta, \phi) \in Z^N$

$$\begin{aligned}
 \langle A^{NN}, z^N \rangle_g &= \langle A_0 \eta + A_1 \phi^N(-r) + \int_{-r}^0 A^N(\theta) \phi^N(\theta) d\theta, \eta \rangle \\
 &\quad + \frac{1}{2} |\eta|^2 - \frac{1}{4} |\phi^N(-r)|^2 - \frac{1}{8} \left(\frac{2}{r}\right) \int_{-r}^0 |\phi(\theta)|^2 d\theta \\
 &\leq \left(\frac{3}{4} + |A_0| + |A_1 A_1^T| + 2r \int_{-r}^0 |A(\theta)|^2 d\theta\right) |\eta|^2 - \frac{1}{2} \omega_0 \|z^N\|^2 \\
 &\equiv \alpha |\eta|^2 - \frac{1}{2} \omega_0 \|z^N\|^2,
 \end{aligned}$$

where we used the relation: $2 \langle x, y \rangle \leq |x|^2 + |y|^2$ for $x, y \in \mathbb{R}^n$. It then follows that

$$\left\langle \frac{d}{dt} z^N(t), z^N(t) \right\rangle_g \leq \alpha |z^N(t)|^2 - \frac{1}{2} \omega_0 \|z^N(t)\|^2$$

or

$$\|z^N(t)\|_g^2 + \omega_0 \int_0^t \|z^N(s)\|^2 ds \leq \|z^N(0)\|_g^2 + 2\alpha \int_0^t |z^N(s)|^2 ds$$

for all $t \geq 0$.

Hence we obtain

$$\int_0^\infty \|z^N(s)\|^2 ds \leq \frac{1}{\omega_0} (1 + 2\alpha) \|z\|^2.$$

The corollary now follows from Datko's theorem (see Lemma 7.4 in [5]).

(Q.E.D.)

We are now ready to state the main theorem.

Theorem 5.3: If the elements of $A(\cdot)$ are absolutely continuous on $[-r, 0]$ and the semigroup $\{S(t), t \geq 0\}$ is uniformly, exponentially stable, then there exist positive constants ω and M such that, for N sufficiently large

$$\|e^{A^N t}\| \leq M e^{-\omega t}.$$

Proof: It follows from Corollary 3.3 and Lemma 3.4 that there exists an integer N_0 such that, for $N \geq N_0$ $\det \Delta^N(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}^+$. But for $|\omega| \geq |A_0| + |A_1| + C = \beta$ and $\omega \in \mathbb{R}$, it follows from Corollary 3.3 that

$$|\Delta^N(i\omega)^{-1}| \leq \frac{1}{|\omega| - \beta}.$$

This inequality, when combined with Lemma 3.4 shows that there exists a constant α such that for $N \geq N_0$

$$|\Delta^N(i\omega)^{-1}| \leq \alpha, \quad \omega \in \mathbb{R}.$$

Note that for $z \in Z$

$$e^{A^N t} z = e^{A_0^N t} z + \int_0^t e^{A^N(t-s)} I^* f(s) ds \quad (5.4)$$

where $f(t) = F e^{A_0^N t} z$, $t \geq 0$ and F is an operator on Z defined by

$$F(\eta, \phi) = (I + A_0)\eta + A_1\phi(-r) + \int_{-r}^0 A(\theta)\phi(\theta)d\theta \quad \text{for } (\eta, \phi) \in Z.$$

It then follows from Lemma 5.1 that there exists a positive constant γ such that

$$\int_0^{\infty} |I e^{A_N^t} z|^2 dt \quad \text{and} \quad \int_0^{\infty} |f(t)|^2 dt \leq \gamma \|z\|^2.$$

If $y(t) = \int_0^t I e^{A_N(t-s)} I^* f(s) ds$, then the Fourier transformation $\hat{y}(i\omega)$ of y is given by

$$\hat{y}(i\omega) = \Delta^N(i\omega)^{-1} \hat{f}(i\omega).$$

Hence, by the Parseval's equality.

$$\begin{aligned} \int_0^{\infty} |y(t)|^2 dt &= \int_{-\infty}^{\infty} |\hat{y}(i\omega)|^2 d\omega \\ &\leq \alpha^2 \int_{-\infty}^{\infty} |\hat{f}(i\omega)|^2 d\omega = \alpha^2 \int_0^{\infty} |f(t)|^2 dt. \end{aligned}$$

It now follows from (5.4) that

$$\int_0^{\infty} |I e^{A_N^t} z|^2 dt \leq 2\gamma(1 + \alpha^2) \|z\|^2$$

which completes the proof along with Corollary 5.2.

(Q.E.D.)

The following theorem is an important consequence of Theorem 5.3.

Theorem 5.4. If the elements of $A(\cdot)$ are absolutely continuous on $[-r, 0]$, rank $C = n$ and (A, B) is stabilizable, then for N sufficiently large

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Remark: In the above proof the condition: $\text{rank } C = n$ is only used for the derivation of (5.5). But it seems to be enough to assume the detectability of (C,A) for such a derivation.

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